

Numbers to Keep in Mind

- $R_{\odot} \sim 7 \times 10^{10} \text{ cm}$
- $M_{\odot} \sim 2 \times 10^{33} \text{ gm}$
- $L_{\odot} \sim 4 \times 10^{33} \text{ ergs/sec}$
- $T_{\text{eff}} \odot \sim 5780^{\circ}$
- $X \sim 0.75$
- $Y \sim 0.23$
- $Z \sim 0.02$
- $1 \text{ A.U.} \sim 1.5 \times 10^{13} \text{ cm}$
- $1 \text{ pc} = 3.1 \times 10^{18} \text{ cm} = 206265 \text{ A.U.}$
- $M_{\odot}(\text{bol}) = +4.75$
- $1 \text{ km/s} = 4.74 \text{ arcsec/year at } 1 \text{ pc}$

Determining Stellar Luminosities

The only direct way of measuring the luminosities of stars is through the application of geometry, and the inverse square law of light. There are several methods to do this:

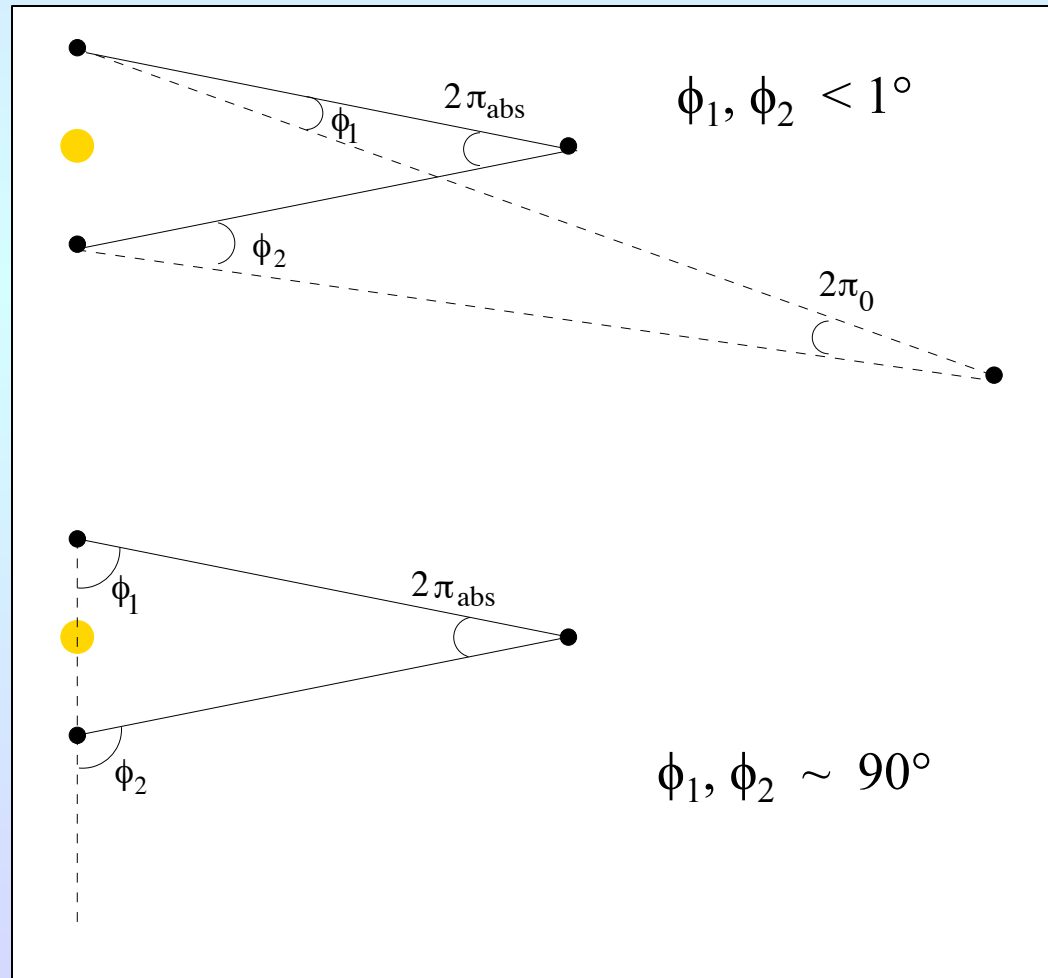
- Parallax
- Moving Clusters
- [Statistical and Secular Parallax]
- [Expansion Parallax, Light Echoes, Baade-Wesselink]

Every other method of determining distances is secondary (i.e., it relies on knowing the luminosities or distances of objects measured by one of these techniques).

Parallax Measurements

Ground-based observations can only produce relative parallax measurements. To obtain absolute parallaxes, one needs a model for the Galaxy (or a special space mission).

Ground-based parallaxes depend on the distances to the background field stars; space missions such as Hipparcos and Gaia obtain absolute parallaxes by using reference stars $\sim 90^\circ$ away.



Astrometric Catalogs

- 1855: Bonner Durchmusterung (BD)
 - Pre-photographic, $\delta > -23^\circ$, 325,000 stars, $m < 10$
- 1892: Córdoba Durchmusterung (CD)
 - Southern extension to BD, $\delta < -22^\circ$, $m < 10$
- 1896: Cape Photographic (CP)
 - Photographic catalog, $-30^\circ < \delta < -40^\circ$, and $\delta < -52^\circ$, 68000 stars
- 1918: Henry Draper (HD) and Henry Draper Extension (HDE)
 - All sky with spectral types, $m < 9$, 360,000 stars
- 1950: Yale Bright Star (BS or HR)
 - Colors, magnitudes, and parallax for 9,100 stars, $m < 6.5$
- 1963: FK4 and FK5 (Fundamental Catalogs)
 - Reference frame precision parallaxes for 1535 stars

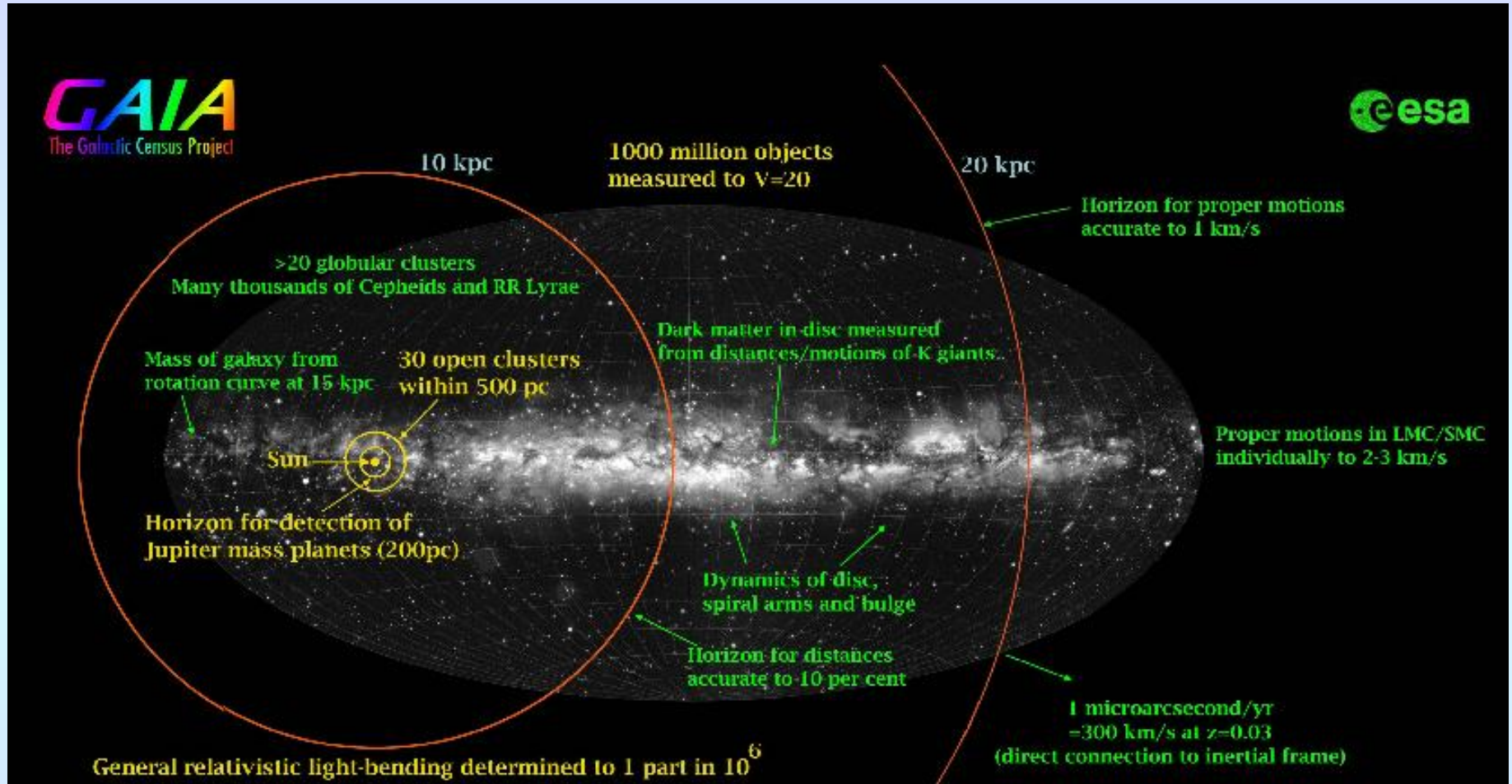
Astrometric Catalogs

- 1990: Hipparcos satellite catalog
 - Absolute precision parallaxes for 100,000 stars, $m < 8$; lower precision data for 2,500,000 other stars to $m < 11$ (Tycho)
- Digitized Sky Survey (I and II)
 - Digitized version of Palomar and UK Schmidt plates; rough magnitudes and astrometry for $\sim 10^9$ objects to $m < 21$
- 2010: USNO-B
 - Digitized version of Palomar and UK Schmidt plates; astrometry and rough magnitudes for $\sim 10^9$ objects to $m < 21$
- 2000: Sloan Digital Sky Survey
 - CCD colors, magnitudes, and positions for $\sim 35\%$ of the sky; photometry to $m < 21$; spectroscopy to $m < 17$ on selected objects
- 2017: GAIA satellite catalog
 - Precision absolute astrometry for $\sim 10^9$ stars

Current Problem with Parallax: Rare Objects

Type of Object	Density	V mag
Open Clusters	5 within 200 pc	3 - 20
Globular Clusters	0 within 2 kpc	---
Cepheid Variables	8 within 500 pc	3 - 6
RR Lyrae Variables	1 within 500 pc	8 - 9
Other Horizontal Branch	~10 within 500 pc	7 - 9
Bright Red Supergiants	0 within 500 pc	---
Bright Blue Supergiants	1 within 500 pc	1
High Latitude Supergiants	2 within 1 kpc	6 - 7
Carbon Stars	13 within 500 pc	5 - 21
Planetary Nebulae	~10 within 500 pc	11 - 16
Cataclysmic Variable	~30 within 350 pc	12 - 16

GAIA Survey



V	10	11	12	13	14	15	16	17	18	19	20	21
σ (μ arcsec)	4.0	4.0	4.2	6.0	9.1	14.3	23.1	38.8	69.7	138	312	1786

For comparison, Hipparcos measured 10% parallaxes to $V \sim 8$

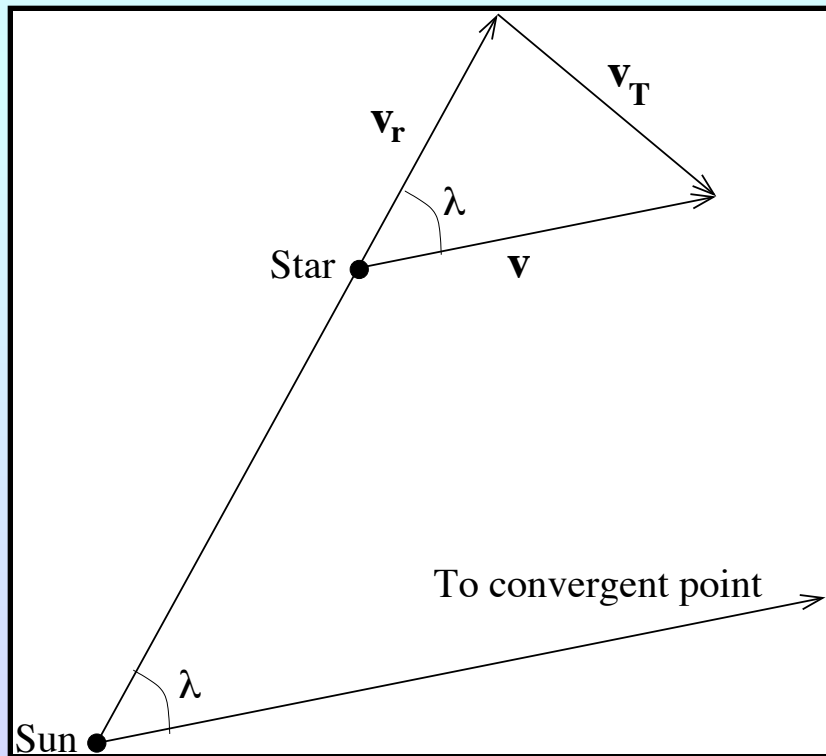
Moving Cluster Method

Premise: For a nearby cluster, both the tangential (proper) motion and radial velocities stars are measurable.

$$v_T = \mu D$$

$$v_r = v_r$$

The ratio of these two quantities can be determined by looking for the convergent point of a cluster.



$$\tan \lambda = \frac{\mu D}{v_r} \Rightarrow D = \frac{v_r \tan \lambda}{\mu}$$

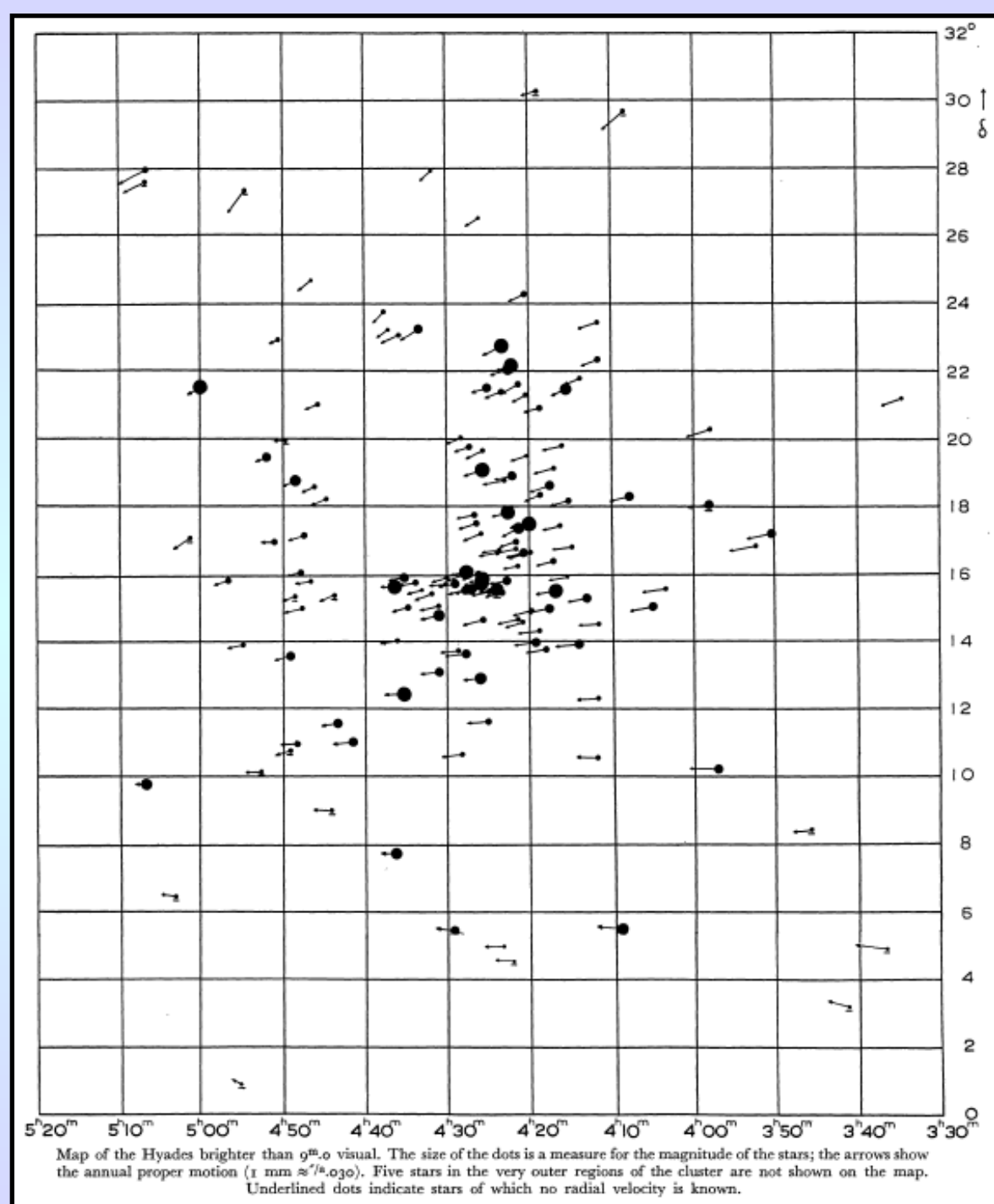
If v_r is in km/s and μ is in arcsec/yr, then

$$D(\text{pc}) = \frac{v_r \tan \lambda}{4.74 \mu}$$

The Hyades

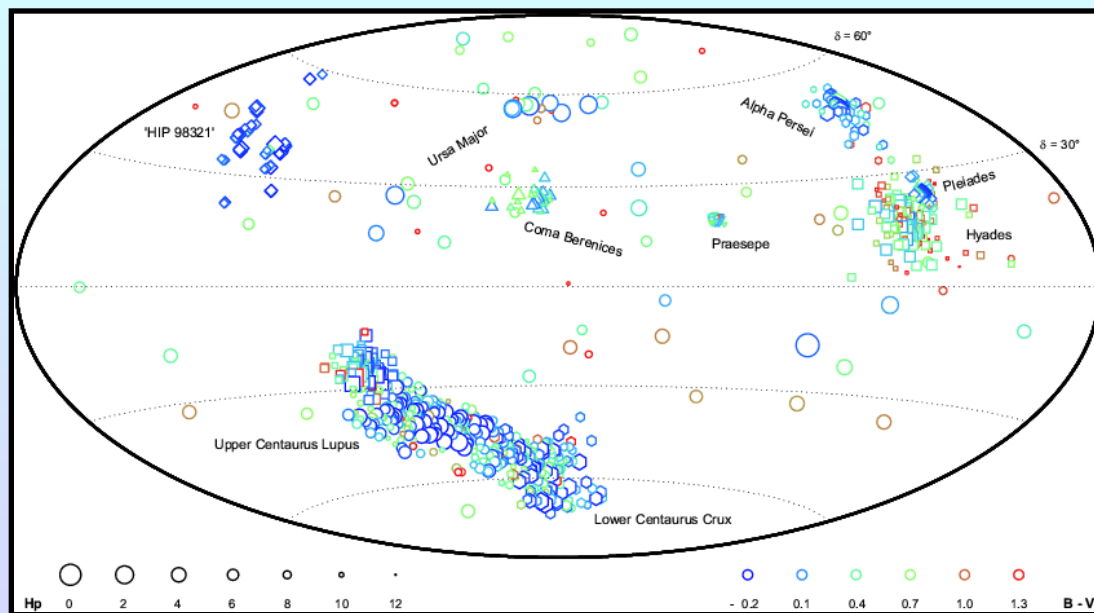
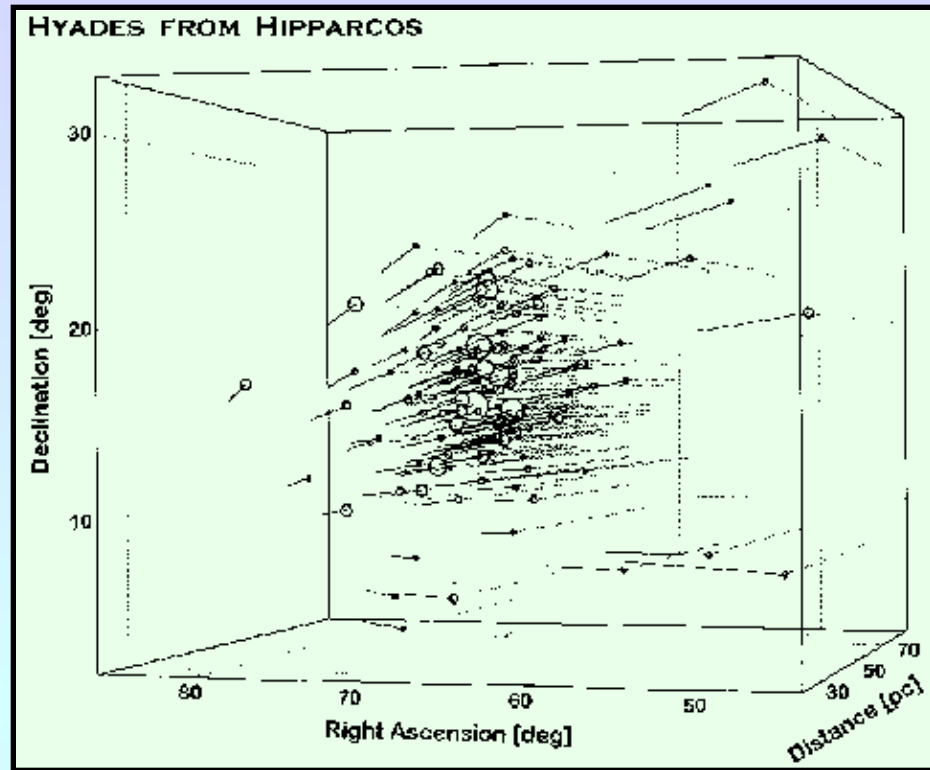
The most famous example of a moving cluster is the Hyades.

[van Bueren 1952, Bull. Astro. Inst. Neth, XI 432, 390]



Moving Cluster Method

Moving cluster distances are available for a few nearby star clusters, such as (famously) the Hyades ($D \sim 45 \pm 1$ pc.) The diagram below shows all the clusters done by the Hipparcos satellite.



Stellar Temperatures

Since stars are not solid bodies, their “size” and “temperature” are somewhat ill-defined quantities. (The emergent flux from a star comes from different levels of the photosphere, which have different temperatures.) We define the “effective temperature” of a star is through the equation

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4$$

This is close to the $\tau = 2/3$ photospheric temperature; for this class, we will not make a distinction.

Stellar Spectral Energy Distributions

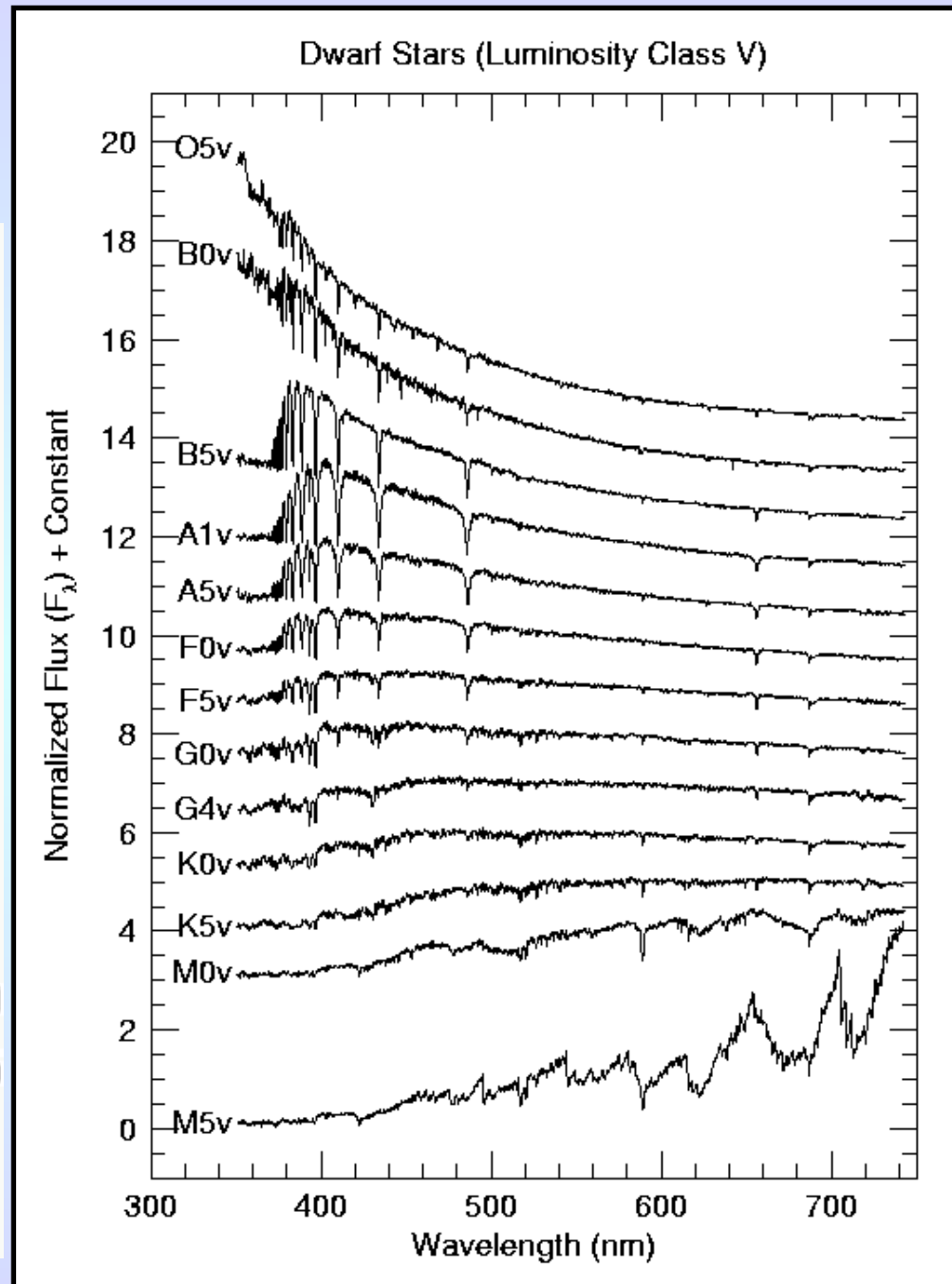
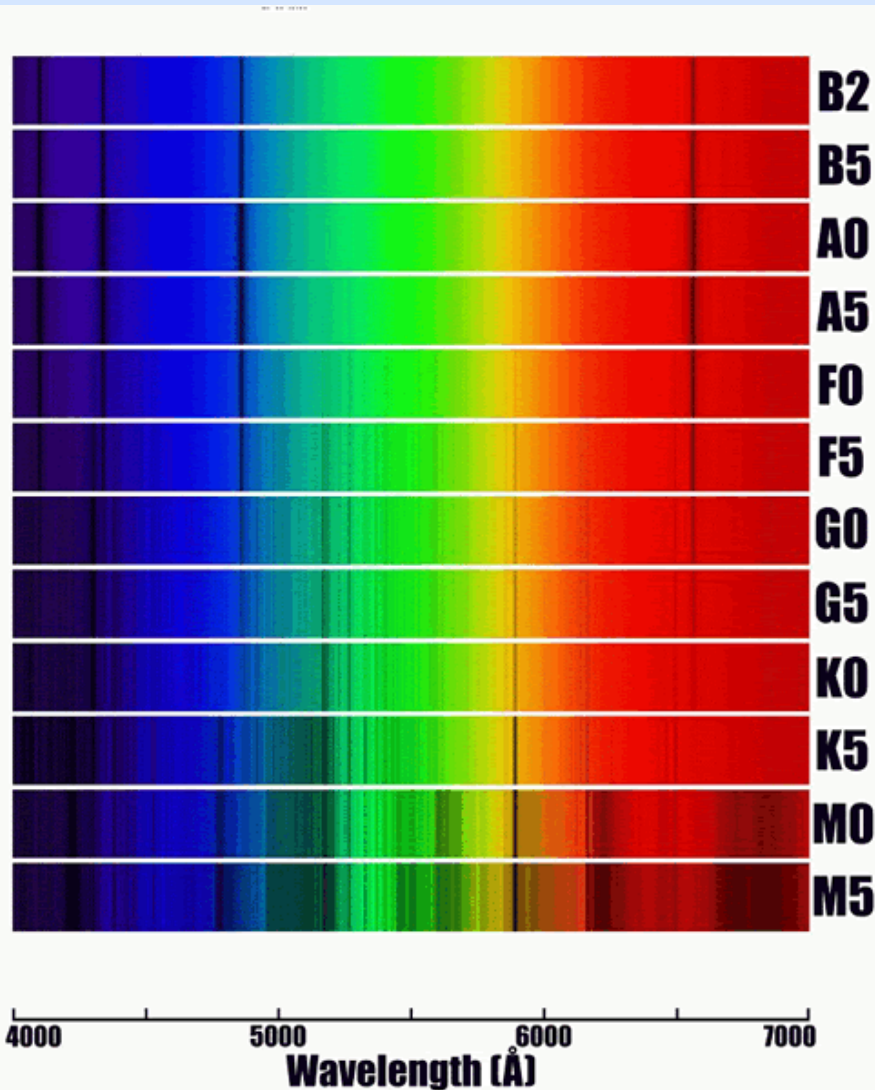
Stellar temperatures range from ~ 3000 K to $\sim 100,000$ K (although there are exceptions). To zeroth order, they can be considered blackbodies, with stellar absorption lines on top.

There is a great variety of stellar absorption lines; the strength of any individual line is determined by the star's

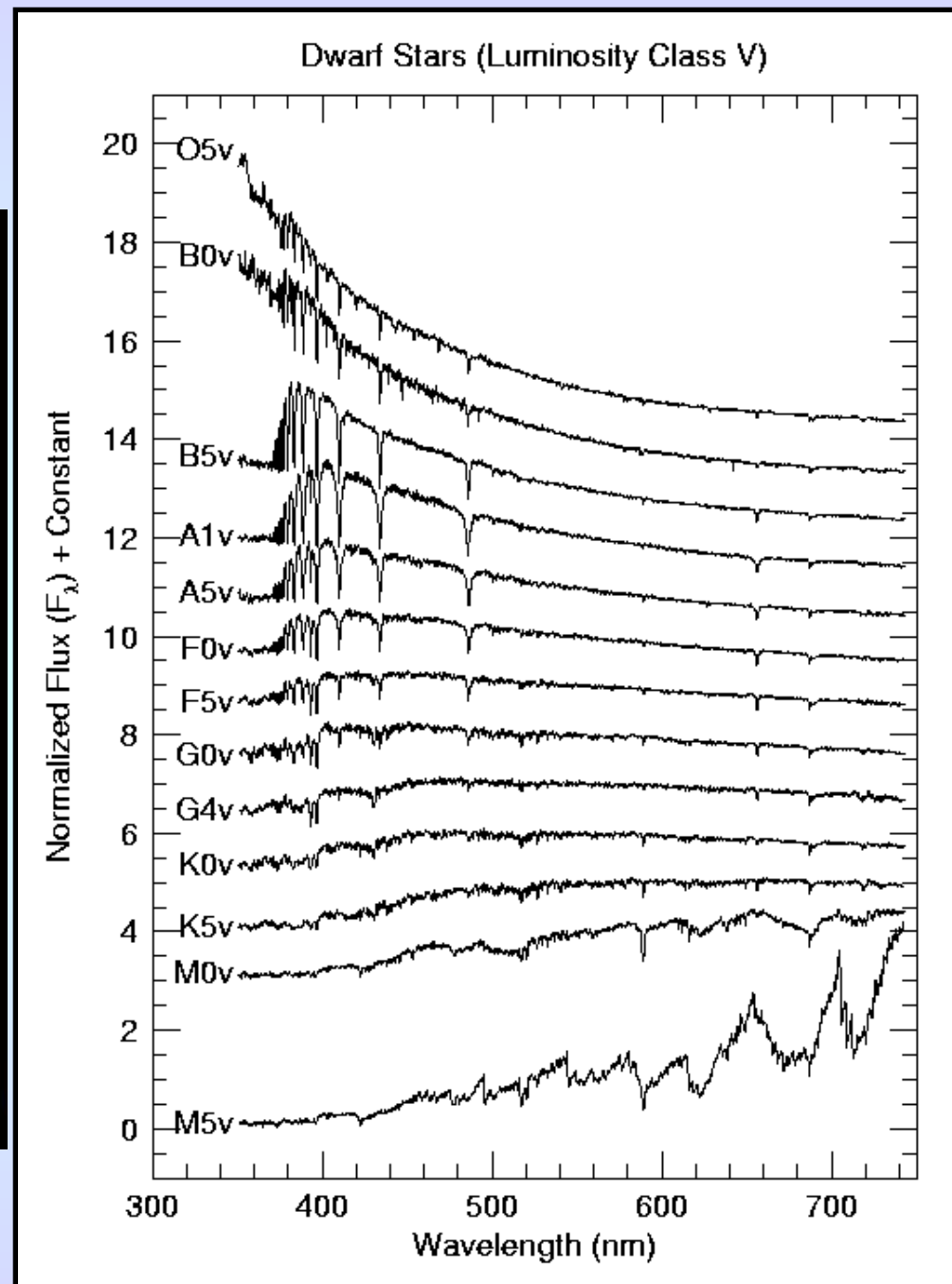
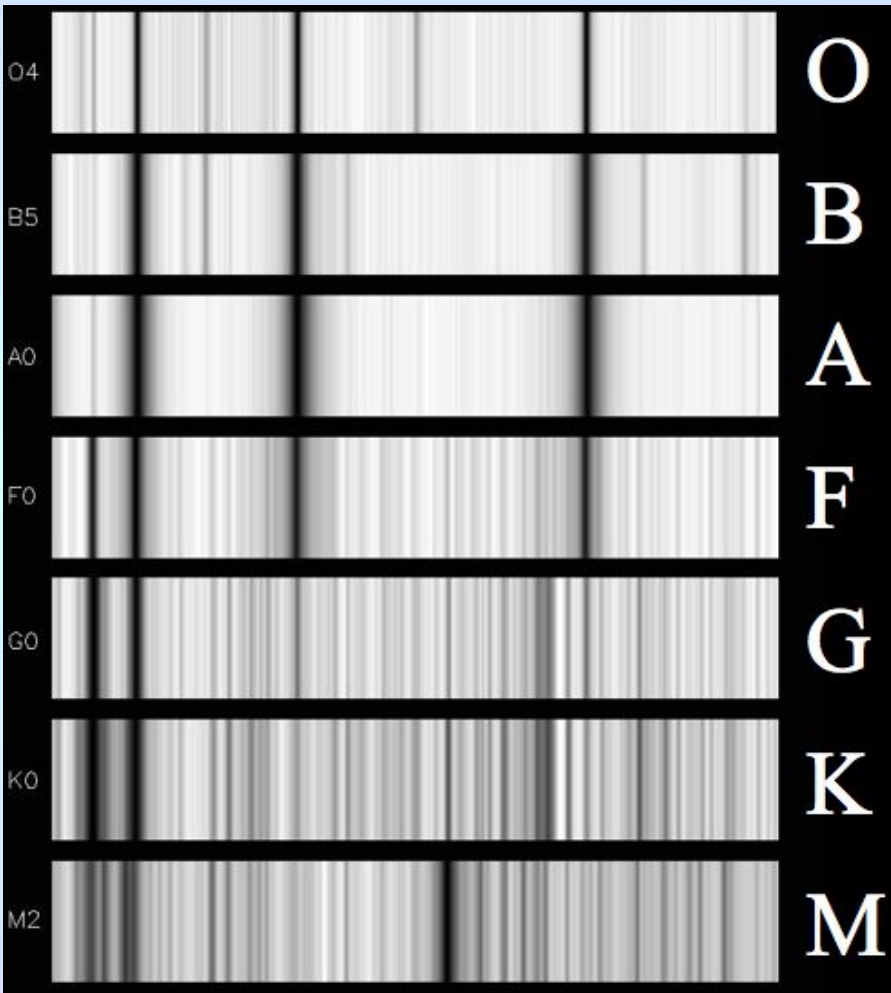
- Temperature (most important)
- Gravity
- Abundance

Historically, stellar spectral types have been classified using letters; the temperature sequence is (hot-to-cool) O-B-A-F-G-K-M-L-T-Y (with numerical subtypes). Orthogonally, stars are also defined in a “luminosity sequence” (which largely reflects size, and is hence a gravity sequence). These are roman numerals, V to I (high-to-low gravity).

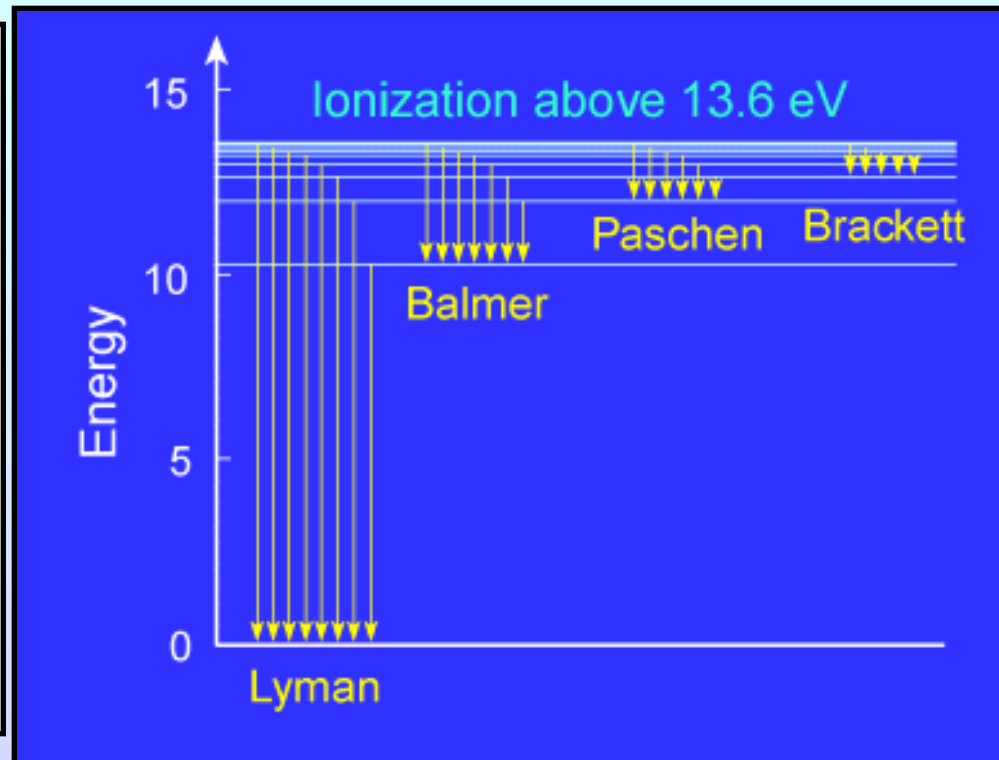
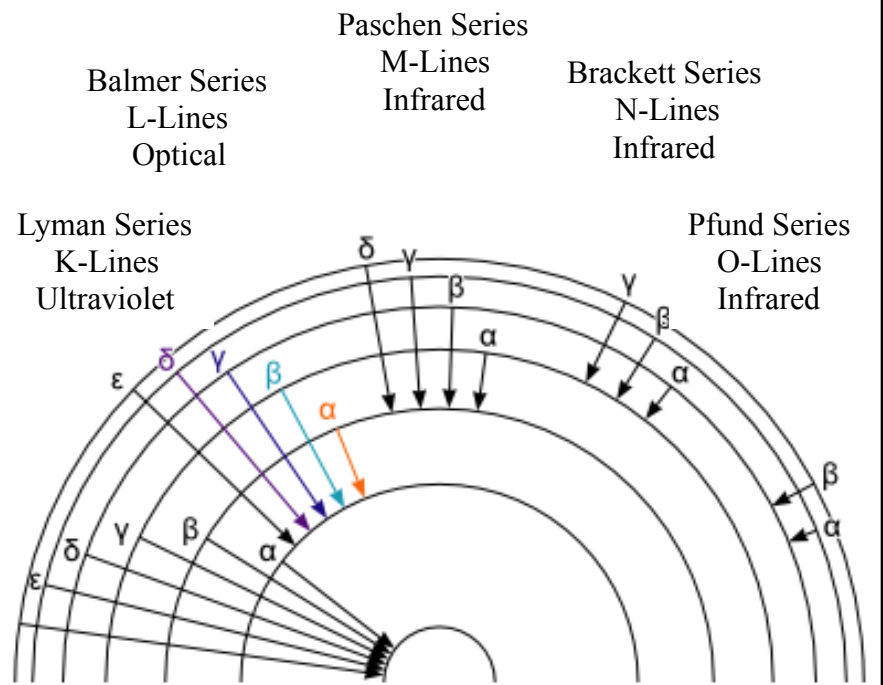
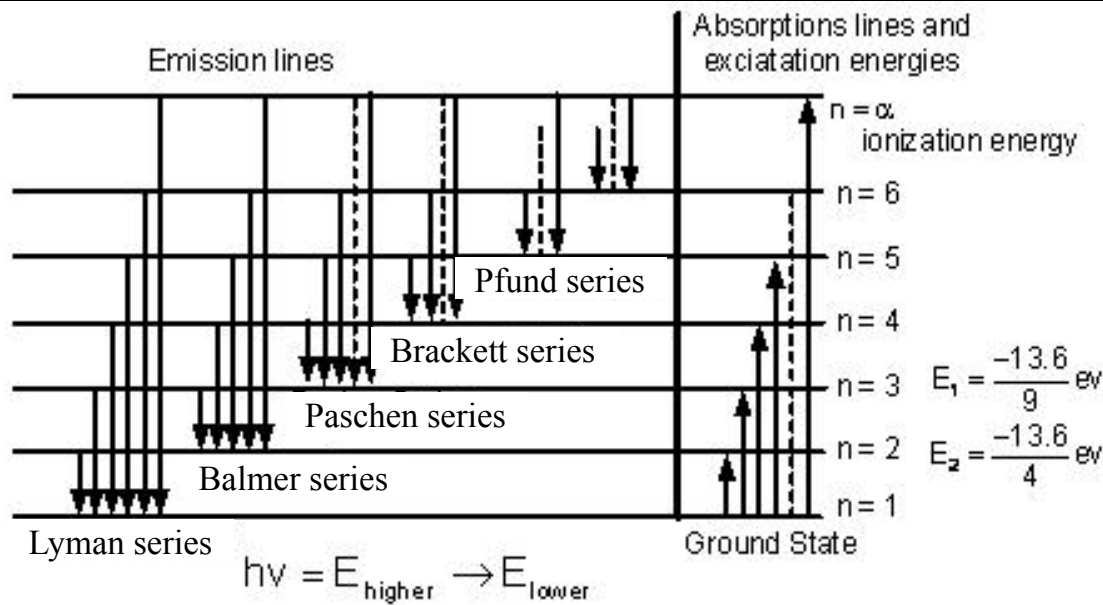
Stellar Spectral Types



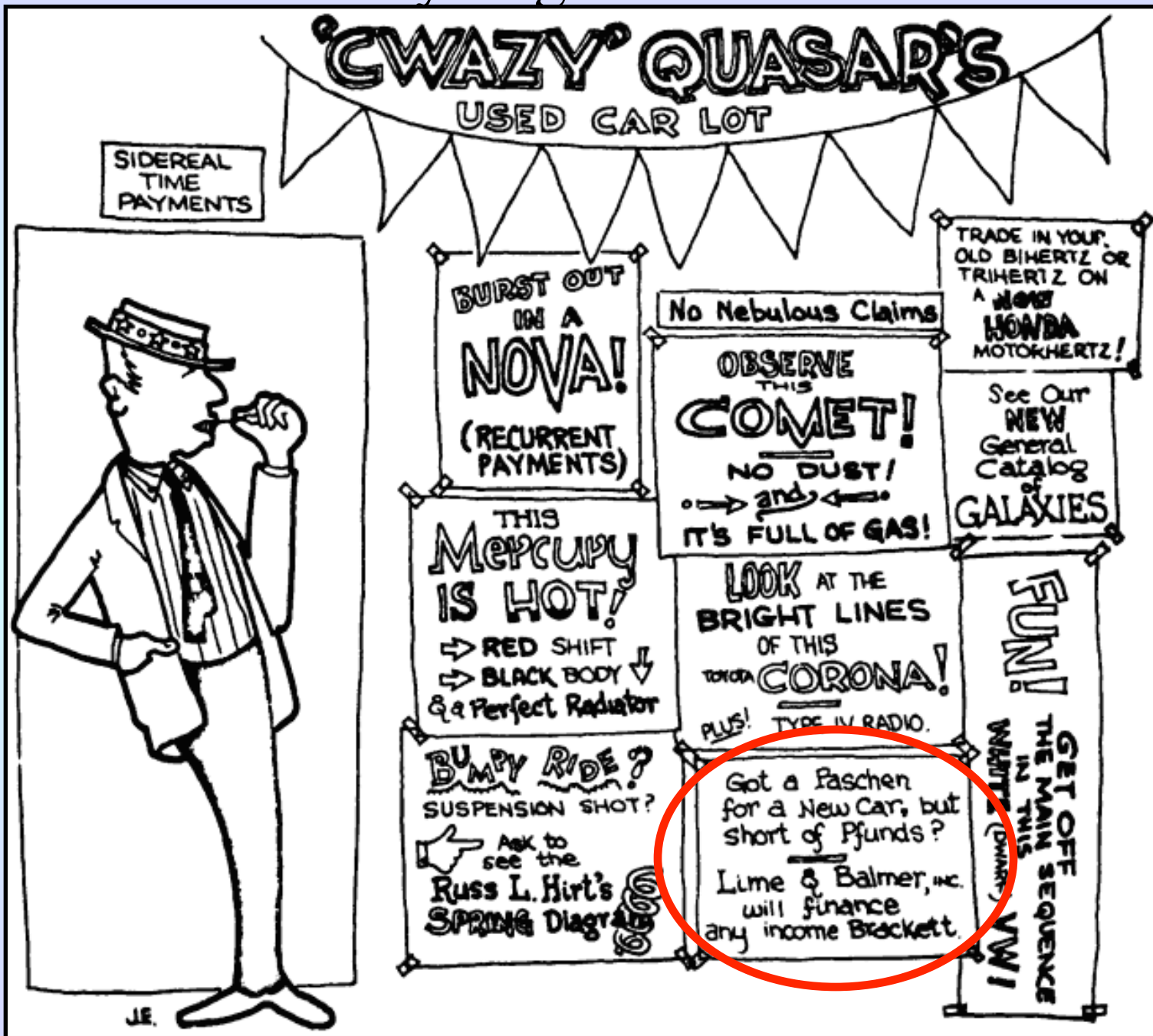
Stellar Spectral Types



Hydrogen Lines



Hydrogen Lines



Saha Equation

The Saha equation gives the fraction of an element in ionization state $i+1$ compared to state i . It requires that a gas be in thermodynamic equilibrium (at least locally), and the densities be low enough so that the mean distance between atoms is larger than the size of an orbital associated with a high energy state.

To derive the Saha equation, we start with the Boltzmann equation, which states that the number of atoms in level i relative to level j is

$$\frac{n_i}{n_j} = \frac{\omega_i}{\omega_j} e^{-\chi_{ij}/k T}$$

where ω_i is the statistical weight of level i (i.e., the number of separate, individual states that have exactly the same energy), and χ_{ij} is the energy difference between the two levels.

Saha Equation

Based on the Boltzmann equation, the number of atoms in level i relative to the number of *all* levels is

$$\frac{n_i}{n} = \frac{\omega_i}{\omega_0 e^{+\chi_{i0}/kT} + \omega_1 e^{+\chi_{i1}/kT} + \omega_2 e^{+\chi_{i2}/kT} + \dots}$$

$$= \frac{\omega_i e^{-\chi_i/kT}}{\omega_0 + \omega_1 e^{-\chi_1/kT} + \omega_2 e^{-\chi_2/kT} + \dots}$$

$$\frac{n_i}{n} = \omega_i \frac{e^{-\chi_i/kT}}{u}$$

where χ_i is the energy difference between the i^{th} level and the ground state, and the variable u is the *partition function* for the atom (or ion). Because u is a function of temperature, it is often written $u(T)$.

Saha Equation

Now let's generalize this equation to electrons in the continuum. Let n_i be the number of atoms in all levels (defined as n above), and let state $i+1$ be that where an excited electron is in the continuum with momentum between p and $p+dp$. The Boltzmann equation then gives

$$\frac{dn_{i+1}}{n_i} = \frac{d\omega_{i+1}}{u_i} \exp \left(-\frac{\chi_i + p^2/2m_e}{kT} \right)$$

where χ_i is the energy needed to ionize the ground state of the atom, and $d\omega_{i+1}$ is the statistical weight of the ionized state. Now consider that $d\omega_{i+1}$ has two components: one from the ion (ω_{i+1}), and the other from the free electron ($d\omega_e$). The former is just the statistical weight of the ground state of the ion, while the latter can be computed using the exclusion rule. Since each quantum cell in phase space can have only two electrons in it (spin up and spin down), then the number of degenerate states in a volume h^3 is

$$d\omega_e = 2 \frac{d^3x d^3p}{h^3} = 2 \frac{dV d^3p}{h^3} = \frac{2}{h^3} dV 4\pi p^2 dp$$

Saha Equation

If we substitute the expression for $d\omega_e$, we have

$$\frac{dn_{i+1}}{n_i} = \frac{8\pi p^2}{h^3} \frac{\omega_{i+1}}{u_i(T)} \exp\left(-\frac{\chi_i + p^2/2m_e}{kT}\right) dV dp$$

Since the number of electrons in volume $dV=1/n_e$, the total number of electrons in all continuum states is therefore

$$\frac{n_{i+1}}{n_i} = \frac{\omega_{i+1}}{u_i(T)} \frac{8\pi}{n_e h^3} e^{-\chi_i/kT} \int_0^\infty p^2 \exp\left(-\frac{p^2}{2m_e kT}\right) dp$$

This integral is relatively easy. If we let $x^2 = p^2/2 m_e k T$, then

Saha Equation

$$\begin{aligned}\frac{n_{i+1}}{n_i} &= \frac{\omega_{i+1}}{u_i(T)} \frac{8\pi}{n_e h^3} e^{-\chi_i/kT} \int_0^\infty (2m_e kT) x^2 e^{-x^2} \cdot (2m_e kT)^{1/2} dx \\&= \frac{\omega_{i+1}}{u_i(T)} \frac{8\pi}{n_e h^3} e^{-\chi_i/kT} (2m_e kT)^{3/2} \int_0^\infty x^2 e^{-x^2} dx \\&= \frac{\omega_{i+1}}{u_i(T)} \frac{8\pi}{n_e h^3} e^{-\chi_i/kT} (2m_e kT)^{3/2} \cdot \frac{\sqrt{\pi}}{4} \\[10pt]\frac{n_{i+1}}{n_i} &= \frac{2}{n_e} \frac{\omega_{i+1}}{u_i(T)} \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_i/kT}\end{aligned}$$

Finally, note that for the calculation above n_{i+1} represents those atoms of species n_i that have one electron in the continuum state, i.e., ionized. It does not consider atoms of n_{i+1} that are themselves excited. (In other words, n_{i+1} only includes ionized atoms in their ground state.)

Saha Equation

To include all the excited states of n_{i+1} , we must again sum the contributions in exactly the same way as we did before. Thus, the statistical weight should be replaced by the partition function, and

$$\frac{n_{i+1}}{n_i} = \frac{2}{n_e} \frac{u_{i+1}(T)}{u_i(T)} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi_i/kT}$$

This is the Saha equation, which relates the number of atoms in ionization state $i+1$ to the number in ionization state i . Note that if need be, we can substitute the electron pressure for the electron density using the ideal gas law ($P_e = n_e k T$), and write the Saha equation as

$$\frac{n_{i+1}}{n_i} P_e = 2 \frac{u_{i+1}(T)}{u_i(T)} \left(\frac{2\pi m_e}{h^2} \right)^{3/2} (k T)^{5/2} e^{-\chi_i/kT}$$

The sense of these equations is intuitive: the higher the temperature, the greater the ratio, but the higher the density (or pressure), the lower the ratio due to the greater possibility for recombinations).

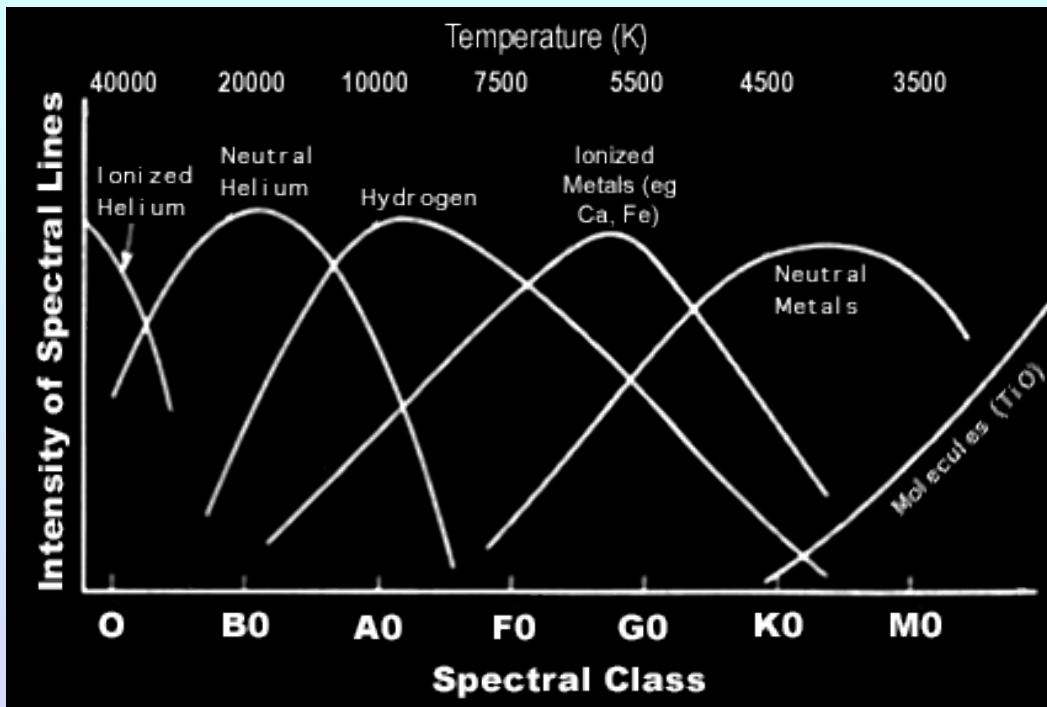
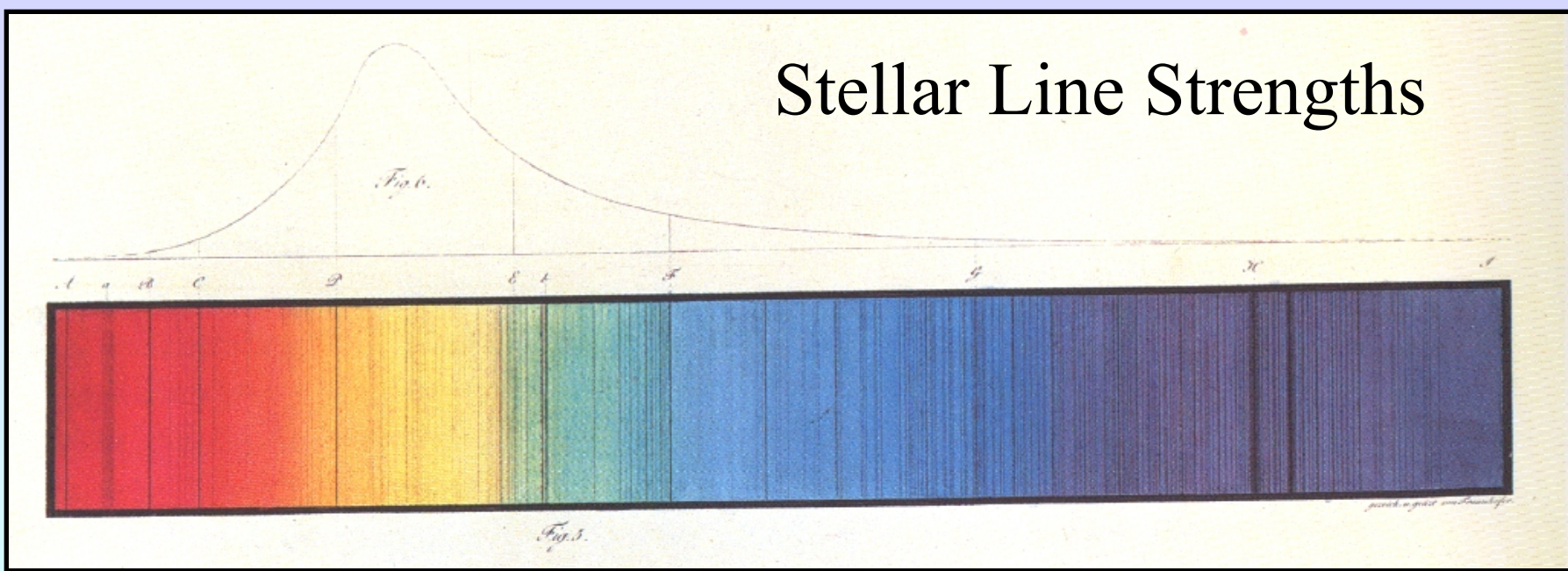
Partition Function for Hydrogen

Values of $\log u(T)$

Θ	T	$\text{Log } P_e = 2$	$\text{Log } P_e = 3$	$\text{Log } P_e = 4$
0.1	50,400	2.77	2.28	1.83
0.14	36,000	2.20	1.63	1.19
0.18	28,000	1.78	1.09	0.75
0.23	21,913	0.92	0.70	0.50
0.3	16,800	0.42	0.35	0.32
0.4	12,600	0.31	0.30	0.30
0.5	10,800	0.30	0.30	0.30

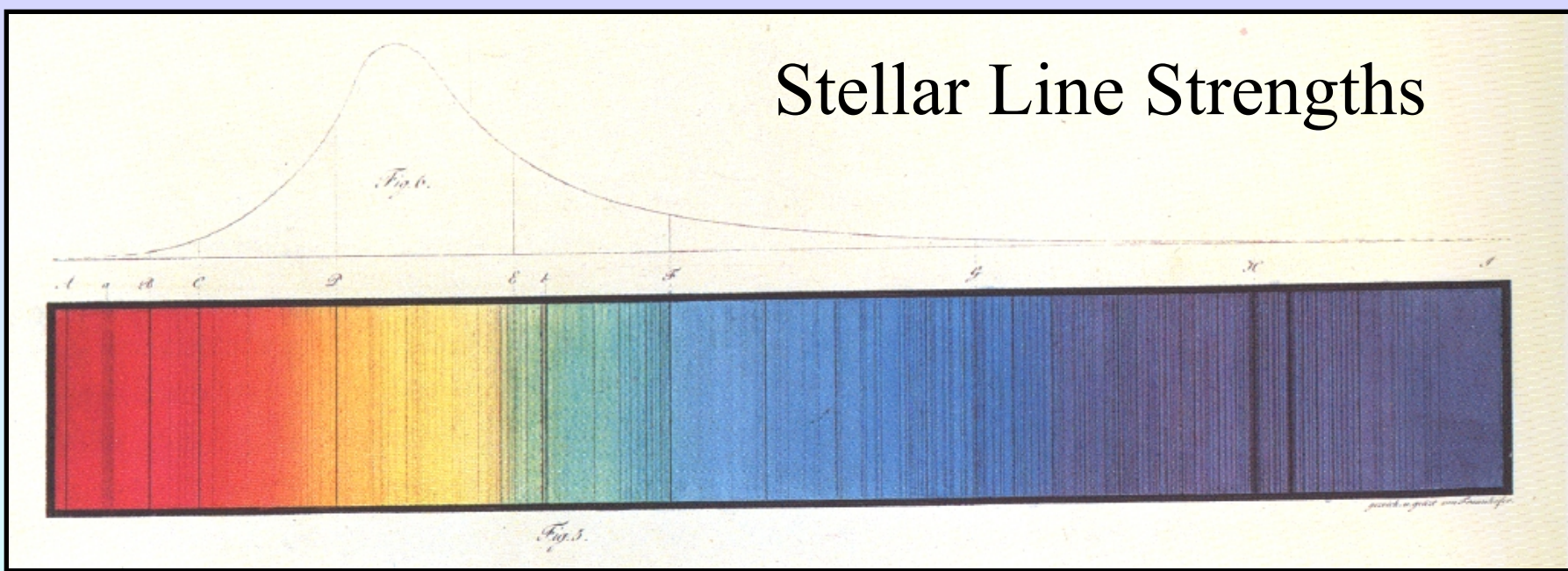
$$\Theta = 5040 / T$$

Stellar Line Strengths



The strength of a given stellar absorption line is largely determined by the element's abundance, in combination with the Saha equation and the Boltzmann distribution.

Stellar Line Strengths

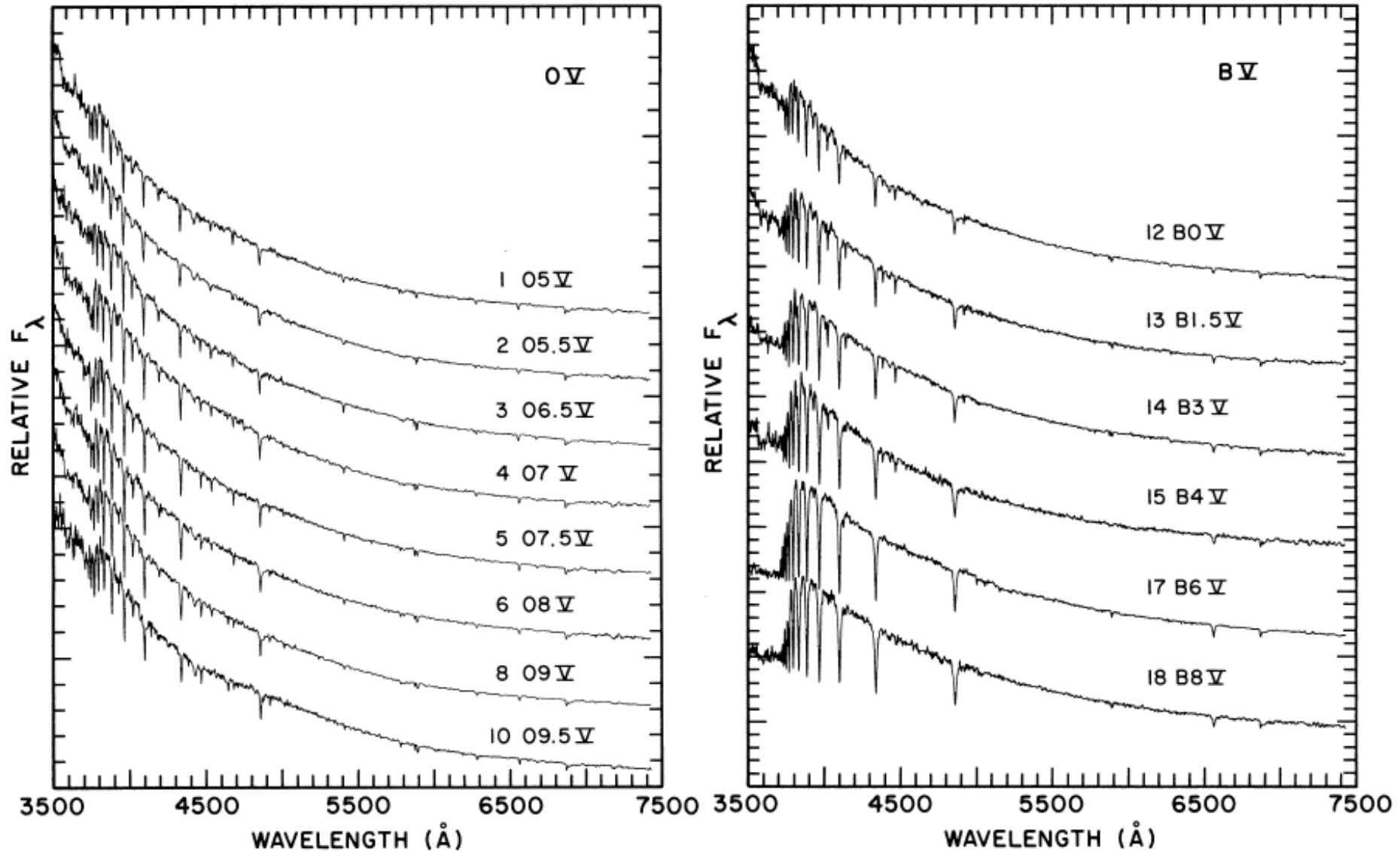


Some of Fraunhofer's names are still with us today

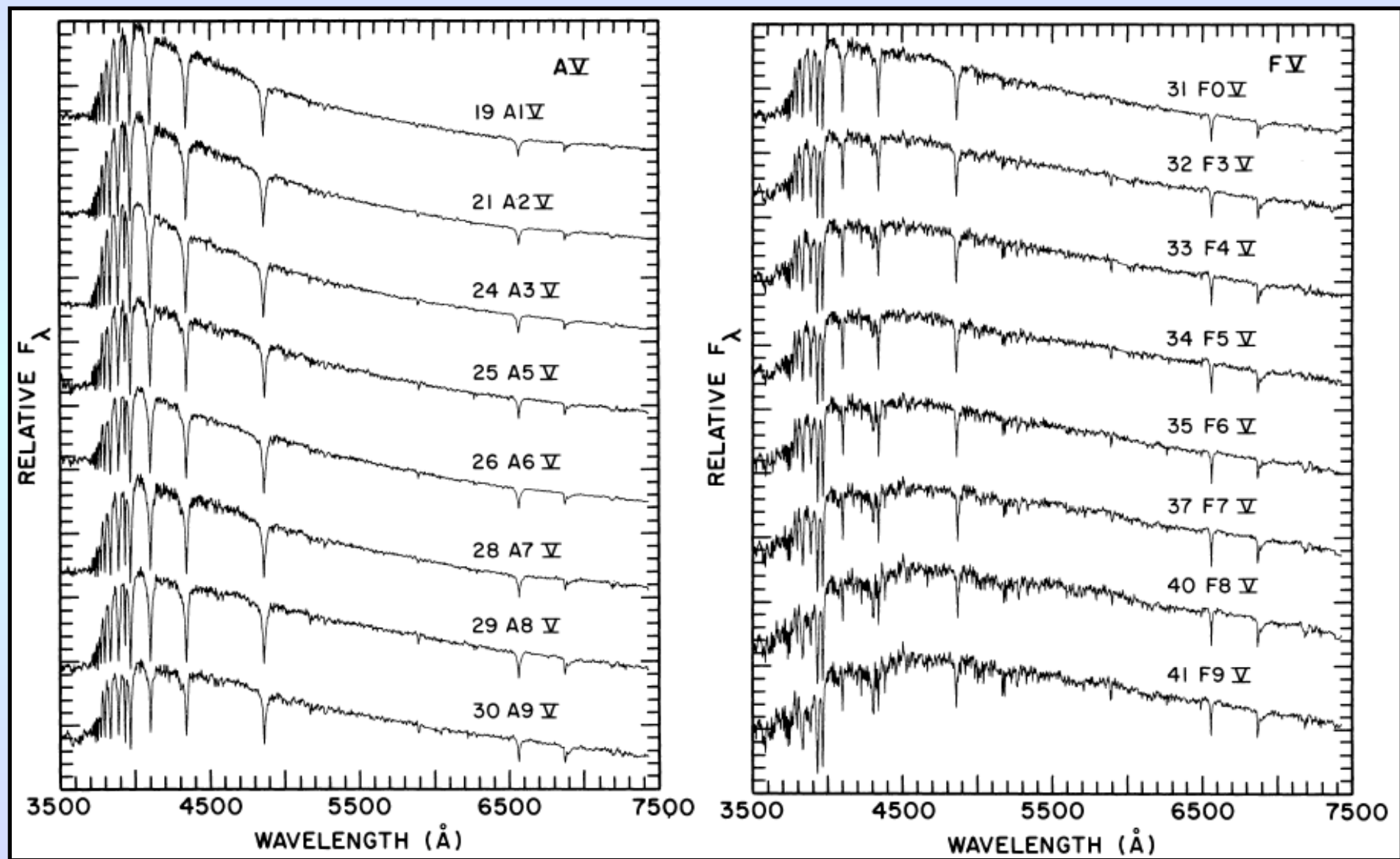
Designation	Wavelength (Å)	Origin
A-band	7600 - 7630	Telluric
B-band	6860 - 6890	Telluric
D-lines	5890, 5896	Sodium
G-band	4295 - 4315	CH complex
H	3968	Calcium
K	3934	Calcium

Stellar Spectral Types: O and B Dwarfs

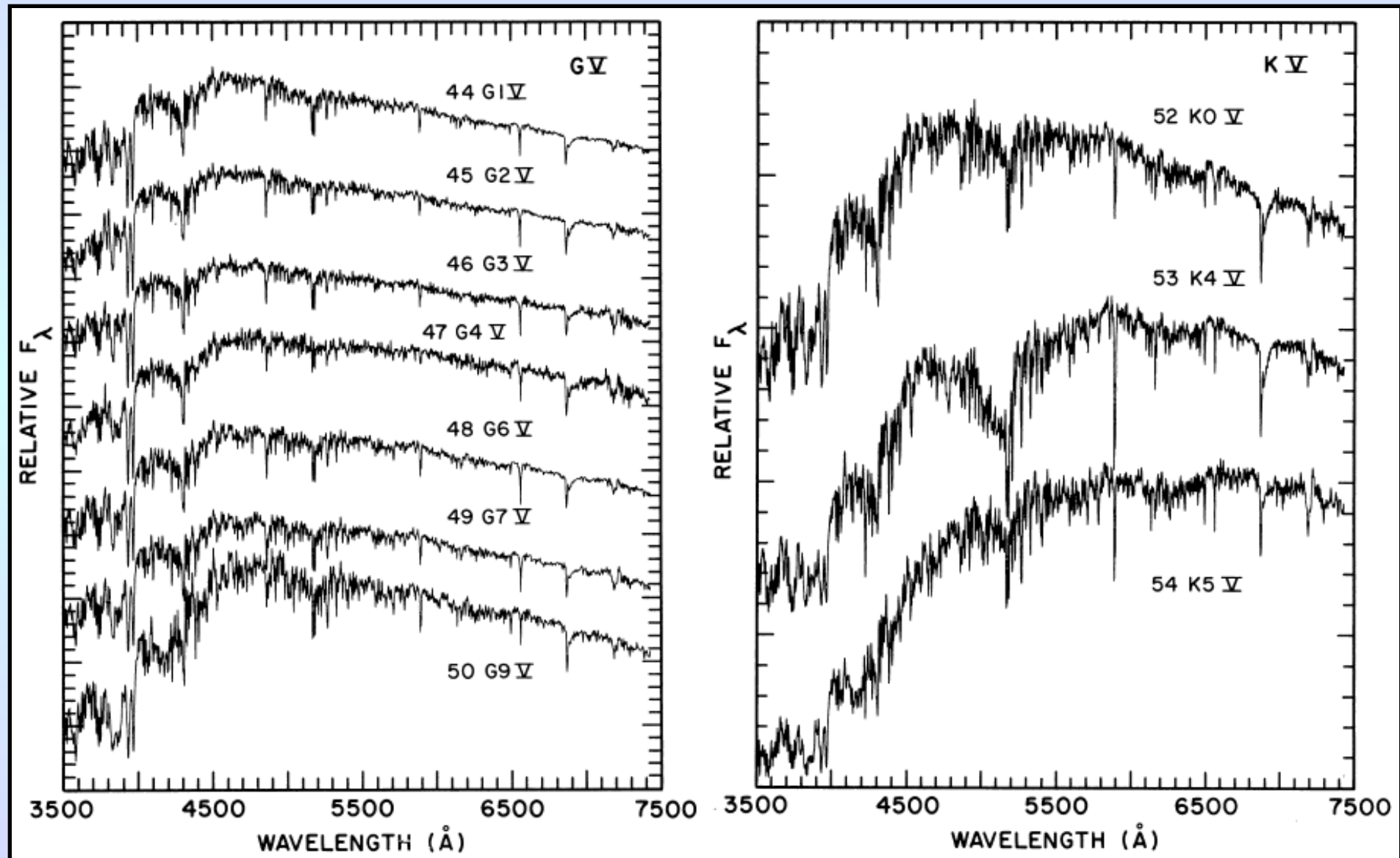
[Jacoby et al. 1984, ApJSupp, 56, 257]



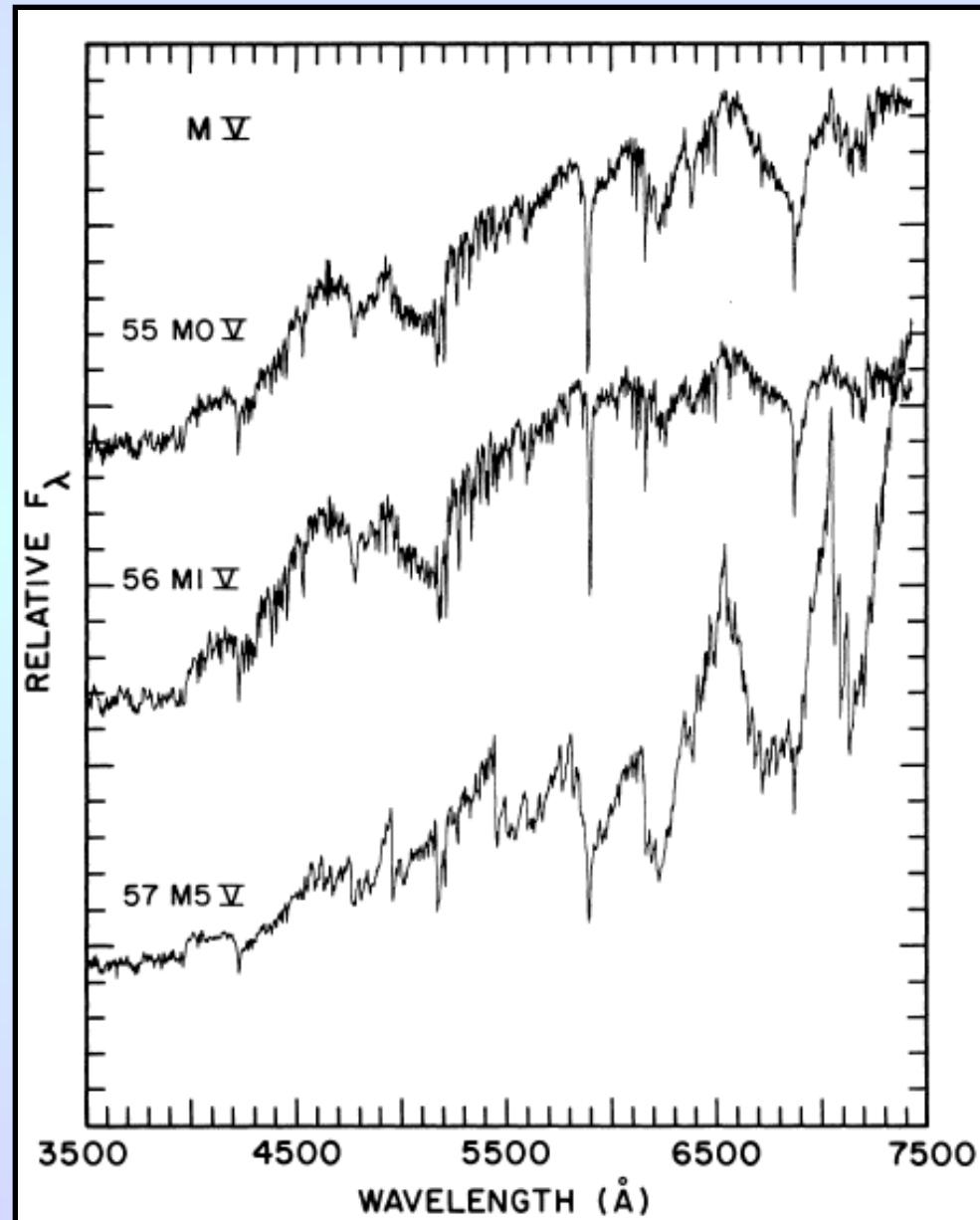
Stellar Spectral Types: A and F Dwarfs



Stellar Spectral Types: G and K Dwarfs

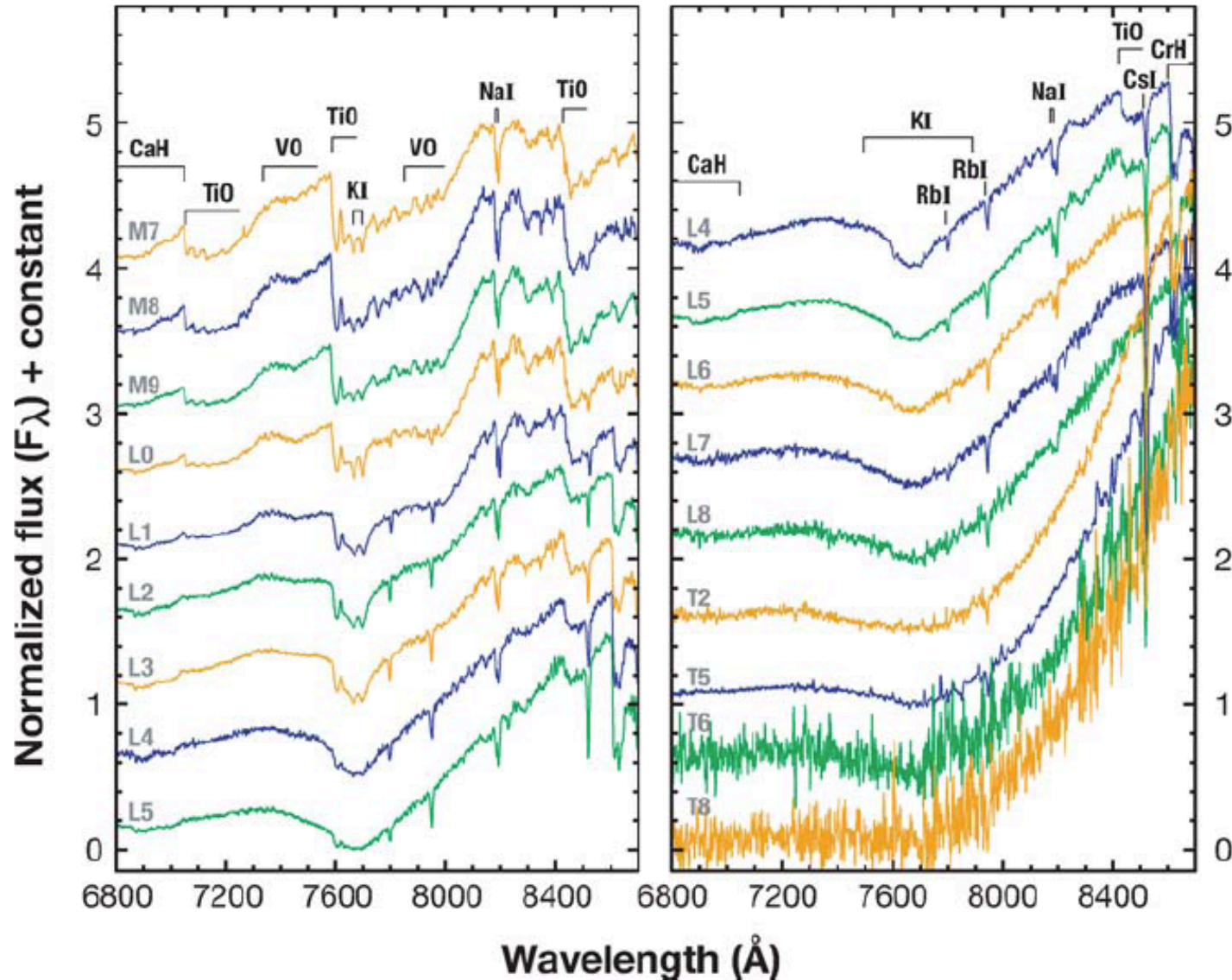


Stellar Spectral Types: M Dwarfs



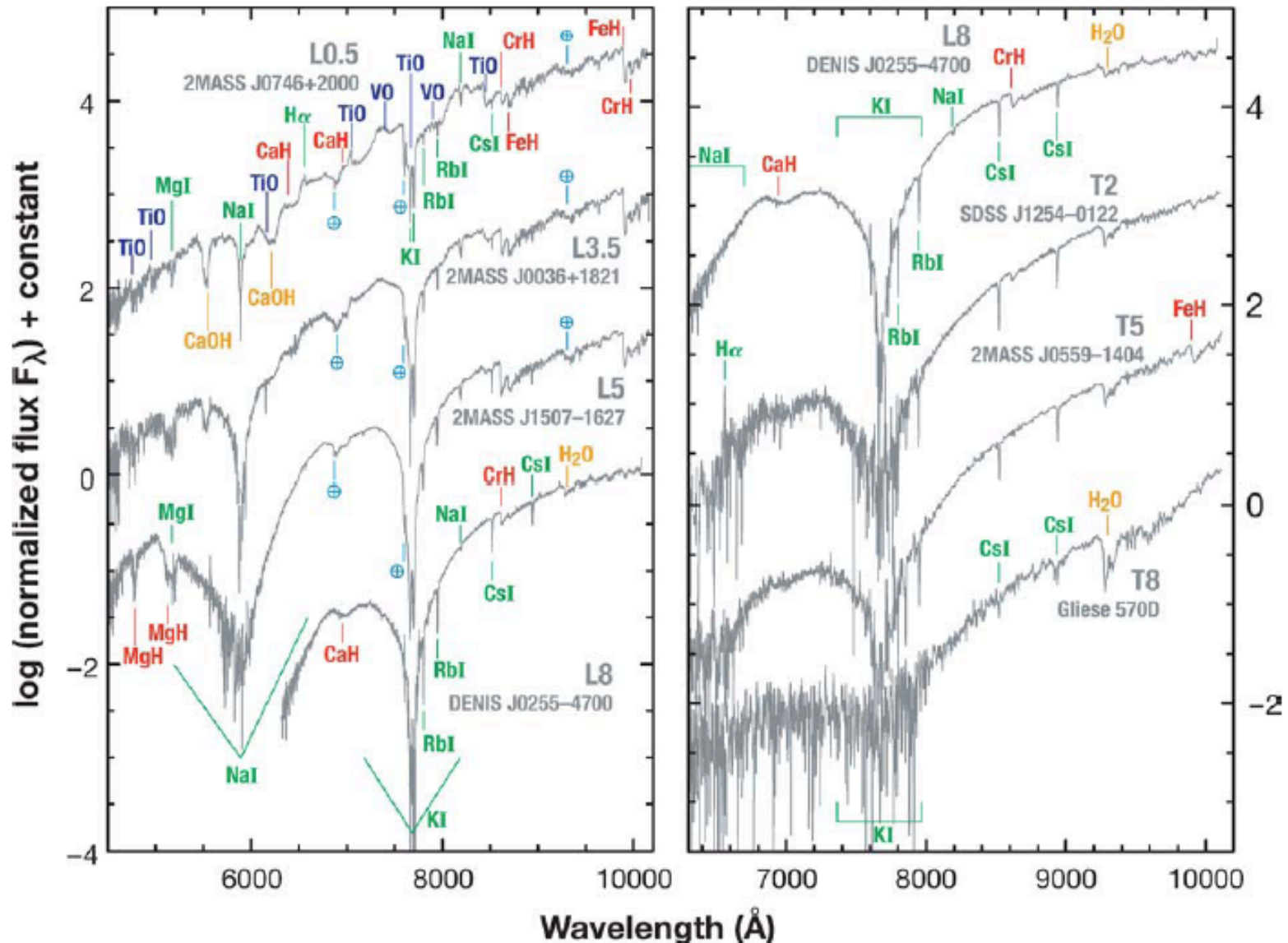
Stellar Spectral Types: L and T Dwarfs

[Kirtpartrick 2005, ARA&A, 43, 195]

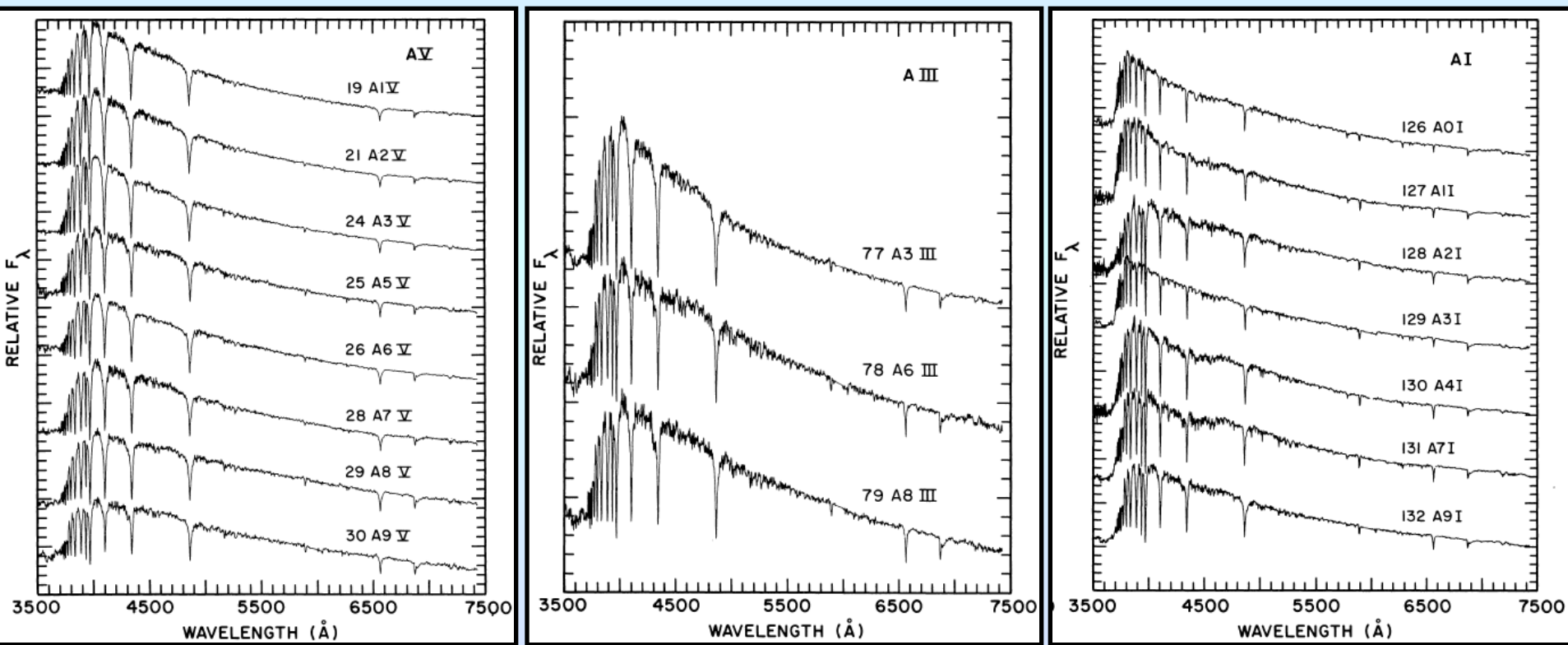


Stellar Spectral Types: L and T Dwarfs

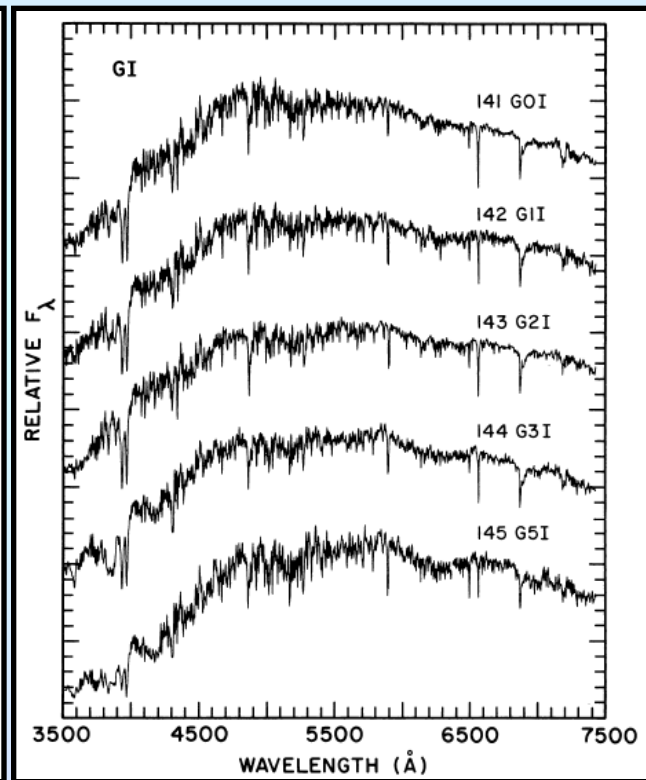
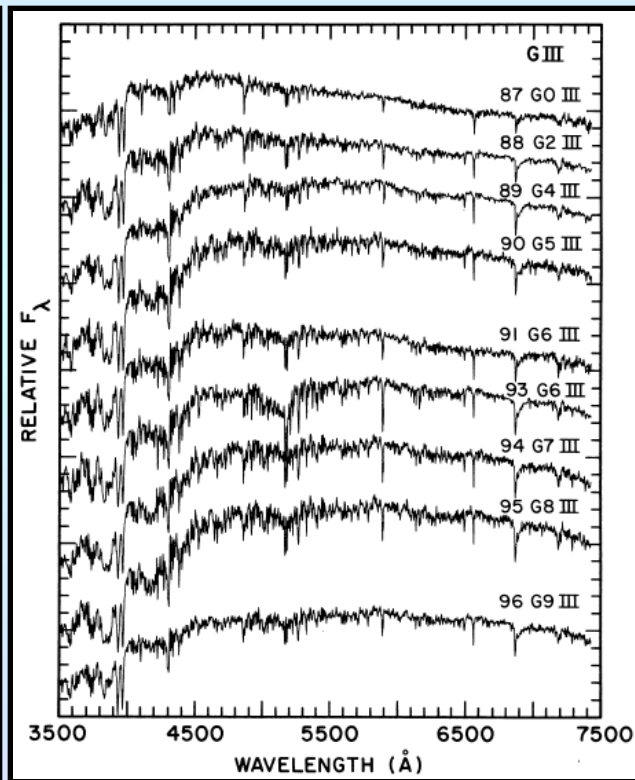
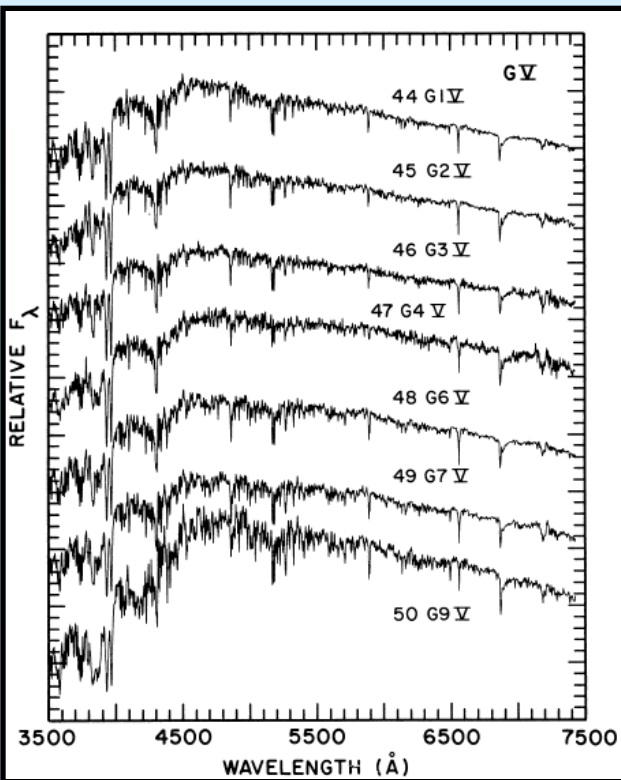
[Kirtpartrick 2005, ARA&A, 43, 195]



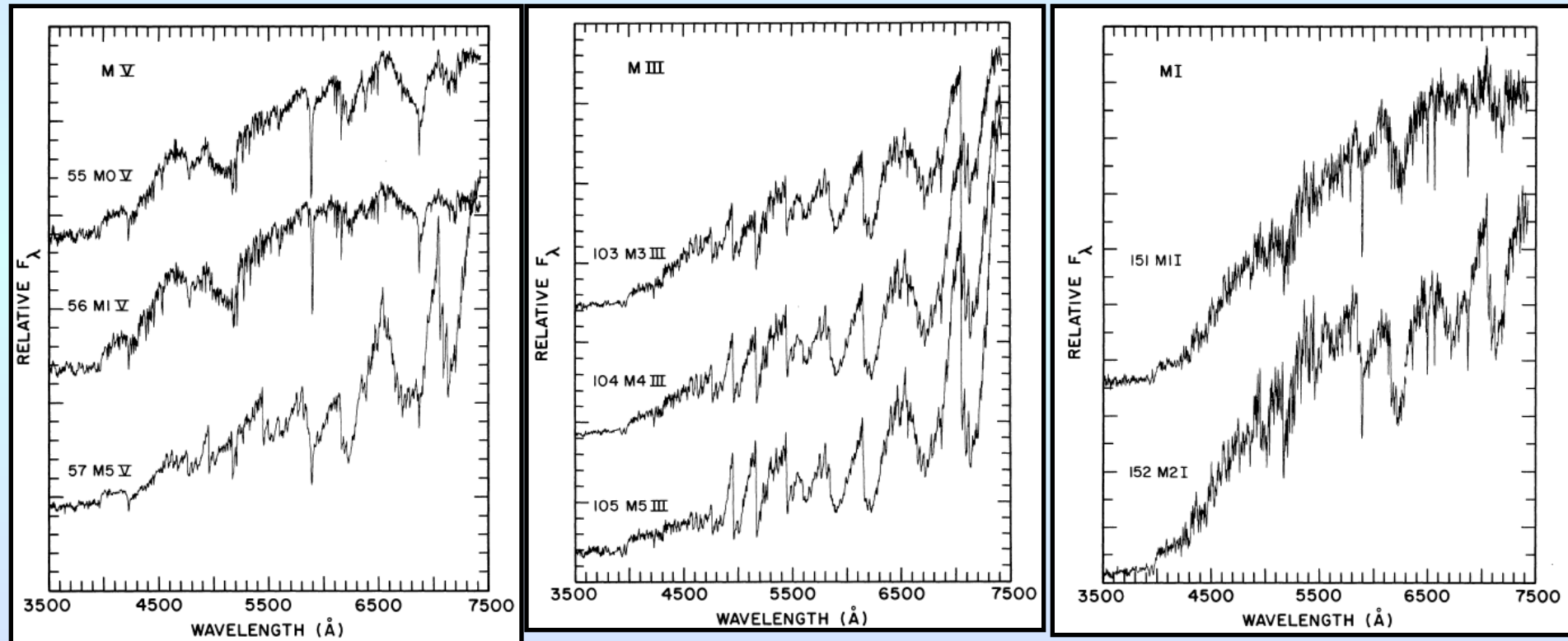
Stellar Spectral Types: A Dwarfs, Giants, and Supergiants



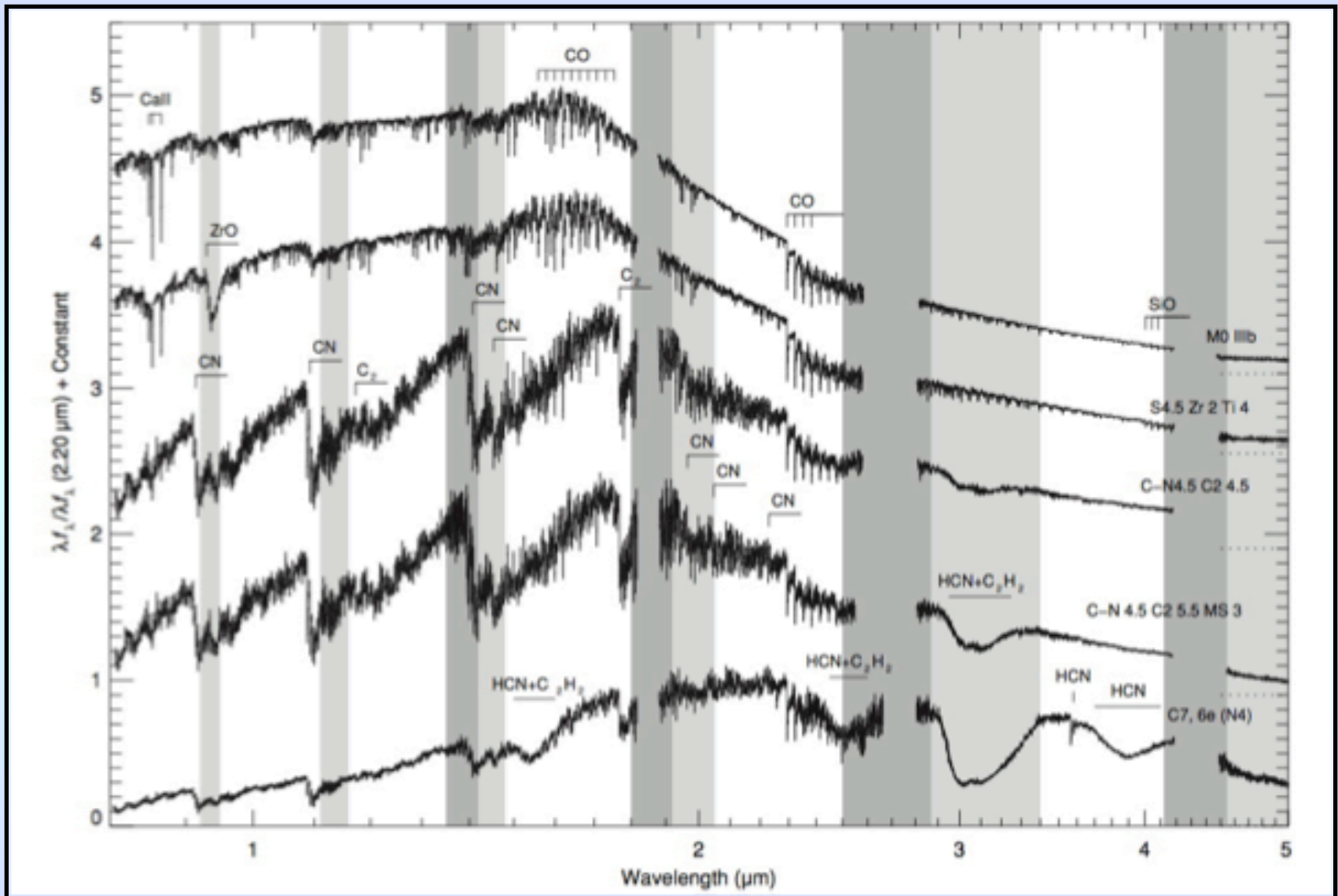
Stellar Spectral Types: G Dwarfs, Giants, and Supergiants



Stellar Spectral Types: M Dwarfs, Giants, and Supergiants



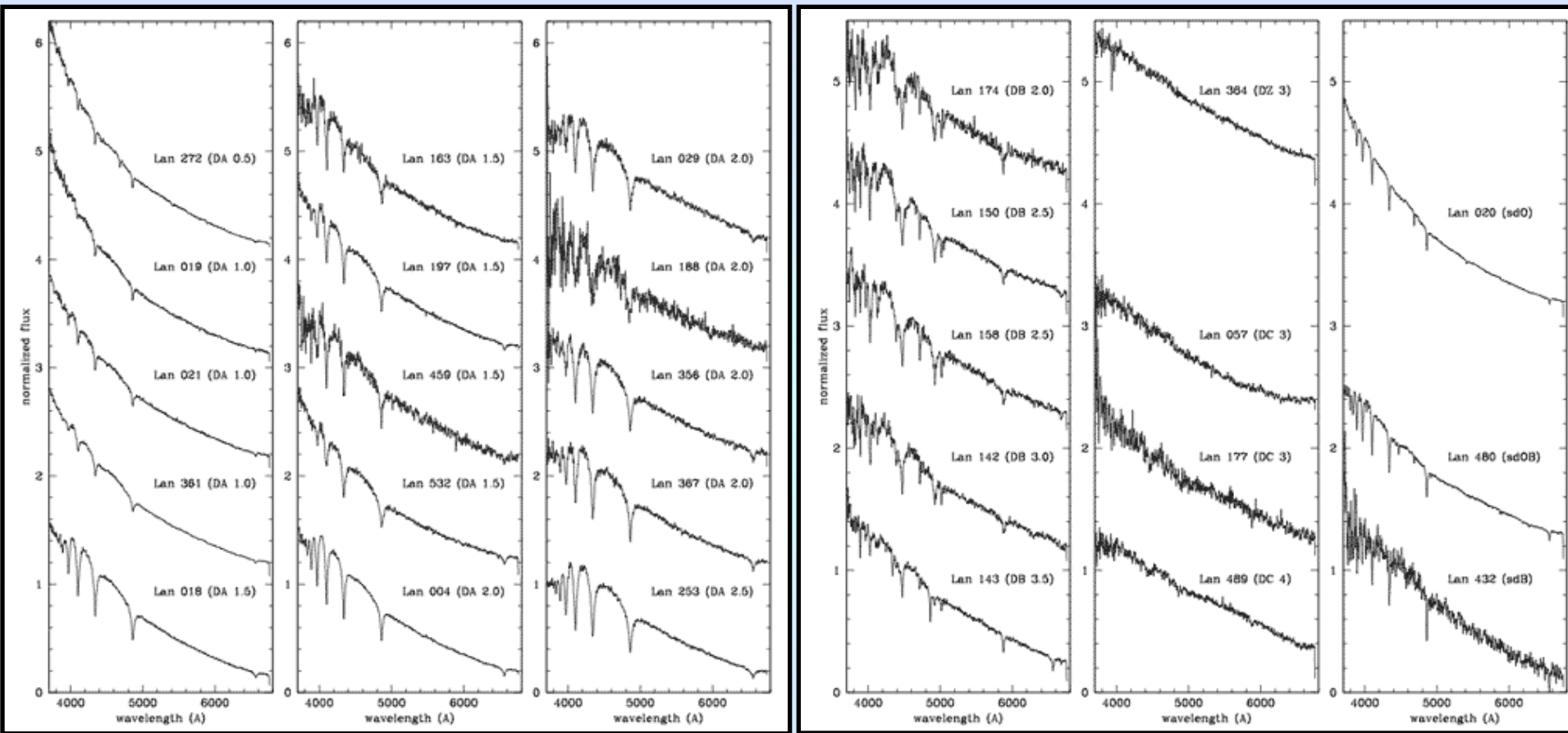
Stellar Spectral Types: M, S, and C Giants



Similar temperature, different C/O ratio

Stellar Spectral Types: DA, DB, and DC

White Dwarfs



DA: Hydrogen absorption, but no helium or metals

DB: Helium absorption, but no hydrogen or metals

DC (or DZ): metal absorption, but no hydrogen or helium

Stellar Spectral Type and Temperature

Main Sequence

Spectral Type	Temperature
O5	40,000
B0	28,000
B5	15,500
A0	9,900
A5	8,500
F0	7,400
F5	6,880
G0	6,030
G5	5,520
K0	4,900
K5	4,130
M0	3,480
M5	2,800
M8	2,400

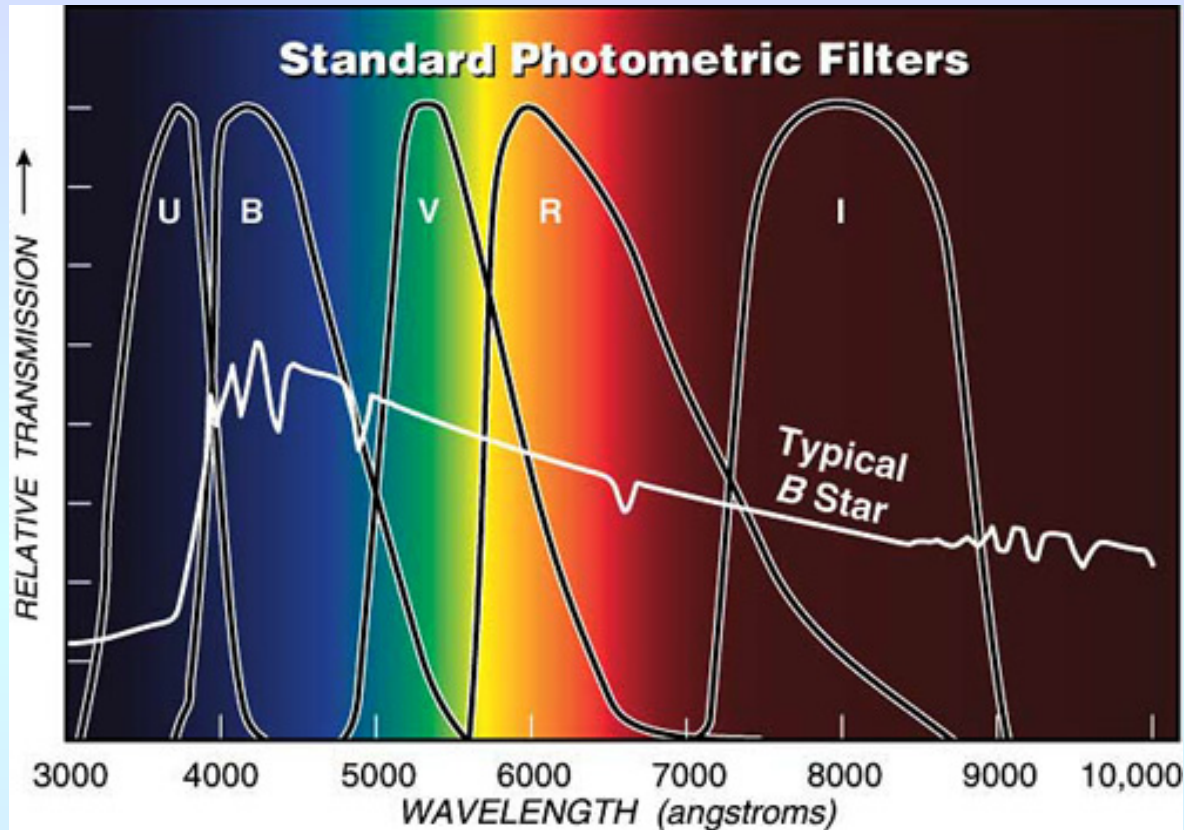
Giants

Spectral Type	Temperature
G0	5,600
G5	5,000
K0	4,500
K5	3,800
M0	3,200

Supergiants

Spectral Type	Temperature
B0	30,000
A0	12,000
F0	7,000
G0	5,700
G5	4,850
K0	4,100
K5	3,500

Color and Temperature



If spectroscopy isn't available (or if the object is too faint), colors can be used to estimate temperature.

Main Sequence

B-V	Temperature
-0.35	40,000
-0.31	28,000
-0.16	15,500
0.00	9,900
+0.13	8,500
+0.27	7,400
+0.42	6,880
+0.58	6,030
+0.70	5,520
+0.89	4,900
+1.18	4,130
+1.45	3,480
+1.63	2,800
+1.80	2,400